Math 140 Business Calculus Final Exam Review Exercises

1. **A**: Find the limit (if it exists) as indicated. **Justify your answer**.

a) $\lim_{x\to 2} x^3 - 5x^2 + 14x - 10$ (Ans: 6)

b) $\lim_{x\to -3} \frac{3x^2+3x-18}{x+3}$ (Ans: -15)

c) $\lim_{x\to 1} \frac{3+2x}{x-1}$ (Ans: DNE)

d) $\lim_{x\to 4} \frac{2-\sqrt{x}}{4-x}$ (ans:1/4)

e) $\lim_{x\to 2} \frac{|x-2|}{x-2}$ (Ans: DNE)

f) $\lim_{x\to -2^+} \frac{|x+2|}{x+2}$ (Ans: 1)

g) $\lim_{x\to 3} \frac{x^2-9}{x-3}$ (Ans: 6)

h) $\lim_{x\to \infty} \frac{2x^3+3x^2-x-1}{4x^3-x^2+3x-5}$ (Ans ½)

I) $\lim_{x\to \infty} \frac{2x^4+3x^2-x-1}{4x^7-x^2+3x-5}$ (Ans: 0)

j) $\lim_{x\to -\infty} \frac{2x^3+3x^2-x-1}{x^2+3x-5}$ (Ans: $-\infty$)

k) $\lim_{x\to -\infty} \frac{10}{e^{-x}+2}$ (Ans: 0)

1. **B**: Prove the Limits

1. $\lim_{x\to 2} x^3 - 5x^2 + 14x - 10 = 6$
2. $\lim_{x\to 3} \frac{x^2-9}{x-3}$ =6
3. $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$ =6
4. $\lim_{x\to 3} \frac{x^2+9}{x-3}$ =DNE
5. $\lim_{x\to \infty} \frac{2x^3+3x^2-x-1}{4x^3-x^2+3x-5} = 0.5$
6. $\lim_{x\to -\infty} \frac{2x^3+3x^2-x-1}{x^2+3x-5} = -\infty$
7. $\lim_{x\to \infty} \frac{10}{e^{-x}+2} = 5$

2. i) Let $f(x) = \begin{cases} 2x^2-1 & if \quad x<2 \\ x & if \quad x \ge 2 \end{cases}$

a. Find $\lim_{x\to 0} f(x)$. If the limit does not exist clearly explain why.
Ans: -1

b. Find $\lim_{x\to 2} f(x)$. If the limit does not exist clearly explain why.
Ans: DNE, because left hand limit and right limits are different.

c. Use the **definition of continuity** to determine if $f(x)$ is continuous at $x=0$.
Ans: continuous

d. Use the **definition of continuity** to determine if $f(x)$ is continuous at $x=2$.
Ans: no

ii) $f(x)=\begin{cases} x & if \quad x<1 \\ 2 & if \quad x=1 \\ x^2 & if \quad 1<x<4 \\ x-1 & if \quad x\ge 4 \end{cases}$

a) Find $\lim_{x\to 0} f(x)=0$, $\lim_{x\to 1} f(x)=1$, $\lim_{x\to 4} f(x)=DNE$**.** If the limit does not exist clearly explain why.

b) Find $f(-2)=-2,\ f\ \ 0\ \ =0,\ f\ \ 1\ \ =2,\ f\ \ 4\ \ =3,\ f\ \ 6\ \ =5$

c) Use the **definition of continuity** to determine if $f(x)$ is continuous at $x=0$, 1, 4.

Ans: at 0 continuous; at 1 discontinuous. At 4 discontinuous

3. Consider the function $f(x)=x^2-x+1$.

a. Find the average rate of change of $f(x)$ between the values $x=3$ and $x=5$**.**
Ans: 7

b. Using the limit definition, find the instantaneous rate of change of $f(x)$ at $x=3$**.**
Ans: 5

c. Using your answer to part (b), find the equation of the tangent line at the point $(3,7)$.
Ans: $y-7=5(x-3)$

4. Find the derivatives of the following functions.

$a)\ f(x)=5x^6+\frac{3}{x^2}-2x+10\pi$

$Ans: 30x^5-\frac{6}{x^3}-2$

$b)\ g(x)=\frac{1}{4}x-3\sqrt[3]{x}+\frac{7}{\sqrt{x}}-1000$

$Ans: \frac{1}{4}-x^{-\frac{2}{3}}$

$c)\ h(x)=x^{-2}+x^{-3}+3$

$Ans: -\frac{2}{x^3}-\frac{3}{x^4}$

d. $y=\frac{\sqrt{x}}{2x+1},\ y'=\frac{-2x+1}{2\sqrt{x}(2x+1)}$

e. $y=(x^2+3x+5)(x^3-2x-1)=(2x+3)(x^3-2x-1)+(x^2+3x+5)(3x^2-2)$

$f)\ y=\left(\frac{1}{x}+3\sqrt{x}\right)(x^3-2x-\sqrt[5]{x})$

$$y'=\left(-\frac{1}{x^2}+\frac{3}{2\sqrt{x}}\right)(x^3-2x-\sqrt[5]{x})+\left(\frac{1}{x}+3\sqrt{x}\right)\left(3x^2-2-\frac{1}{x^{\frac{4}{5}}}\right)$$

g) $y=\frac{x-3}{2x^2+1},\ y'=\frac{-2x^2+1-12x}{(2x^2+1)^2}$

$h)\ \ f(x) = 5x^6 + \dfrac{3}{x^2} - 2e^x + 10$

$f'(x) = 30x^5 - \dfrac{6}{x^3} - 2e^x$

$i)\ g(x) = \dfrac{1}{4}x - 3\ln x^3 + \dfrac{7}{\sqrt{x}} - 1000$

$g'(x) = \dfrac{1}{4} - 9\dfrac{1}{x} - \dfrac{7}{2x^{\frac{3}{2}}}$

$j)\ \ h(x) = x^{-2} + x^{-3} + 3\ln(2x)$

$h'(x) = -2x^{-3} - 3x^{-4} + 3\dfrac{1}{x}$

$k)\ \ g(x) = \dfrac{1}{4}x^8 - 3\ln x^2 + \dfrac{7}{\sqrt{x}} - 10$

$g'(x) = 2x^7 - \dfrac{6}{x} - \dfrac{7}{2x^{3/2}}$

5. a) Let $f(x) = 3x + x^7$, find derivative of $f^{-1}(-9)$

b) Let $f(x) = x^2 - 2x$. Does $f(x)$ has inverse function. For what interval it has

b) inverse function? Find the derivative of $f^{-1}(3) = 1/4$.

Answer: $(1,\infty)$, $f^{-1}(3) = 1/4$

c) 6. a) Use idea of linear approximation to approximate

$\sqrt{26}$ $\quad \sqrt[3]{26.05}$

b) Prove that $\sqrt{1+x} \approx 1 + \dfrac{1}{2}x$ *for x close to 0, and illustrate this approximation by drawing the graphs of* $y = \sqrt{1+x}$ *and* $y = 1 + \dfrac{1}{2}x$ *on the same screen.*

We consider $f(x) = \sqrt{1+x} \Rightarrow f'(x) = \dfrac{1}{2\sqrt{1+x}} \Rightarrow f'(0) = \dfrac{1}{2}$ *, as* $x = a = 0$

Now using linear approximation formula we find

$$f(x) \approx f(a) + f'(a)(x-a) \Rightarrow f(x) \approx f(0) + f'(0)(x-0) \Rightarrow \sqrt{1+x} \approx 1 + \frac{1}{2}x$$

c) Prove that $(1+x)^m \approx 1 + mx$ *for x close to 0 and use this approximation to find approximation of* $(26.95)^{1/3}$

d) Use the linear approximation of $f(x) = \sqrt[3]{x}$ *to approximate the value of* $(26.95)^{1/3}$.

Solution. From $f(x) = \sqrt[3]{x}$ *we have* $f'(x) = \dfrac{1}{3}x^{-2/3}$

7. Derivative Practice Problems:

Problem	Answers
1. $y = 4$	0
2. $y = 1{,}000{,}000$	0
3. $y = 3x + 5$	3
4. $y = 0.72x + 7.554$	0.72
5. $y = 2x^3 + x$	$6x^2 + 1$
6. $y = 4x^2 - 6x + 1$	$8x - 6$
7. $y = 2/x^3$	$-6/x^4$
8. $y = 6/x^4 + 1/x$	$-24/x^5 - 1/x^2$
9. $y = \text{sqrt}(x)$	$1/(2\sqrt{x})$
10. $y = 10x^{22}$	$220x^{21}$
11. $y = e^{0.43x}$	$0.43e^{0.43x}$
12. $y = 4e^{2x}$	$8e^{2x}$
13. $y = e^x$	e^x
14. $y = e^{5x} + 9^4$	$5e^{5x}$
15. $y = \ln(2x)$	$1/x$
16. $y = \ln(3/x)$	$-1/x$
17. $y = \ln(x^2)$	$2/x$
18. $y = (x^2 + 1)(e^x + 4x)$	$(x^2 + 1)e^x + 2x(e^x + 4x)$
19. $y = (3x^2 + 5x)(\ln x)$	$\frac{3x^2 + 5x}{x} + \ln x(6x + 5)$
20. $y = (x^2 + 6x - 1)^{10}$	$10(x^2 + 6x - 1)^9(2x + 6)$
21. $y = (5x^3 - 6x)^9$	$9(5x^3 - 6x)^8(15x - 6)$
22. $y = (3x^4 - x^3 + x)^2$	$2(3x^4 - x^3 + 1)(12x^3 - 3x^2 + 1)$
23. $y = e^{3x+2}$	$3e^{3x+2}$
24. $y = \ln(x^3 + 2x + 5)$	$\frac{3x^2 + 2}{x^3 + 2x + 5}$

8. Integration Practice Problems

Problem	Answer:
1. $\int x^{-99}\,dx$	1. $-\frac{1}{98}x^{-98} + c$
2. $\int \frac{5}{x^3}\,dx$	2. $-\frac{5}{2}x^{-2} + c$
3. $\int \frac{x+1}{x}\,dx$	3. $x + \ln\lvert x\rvert + c$

4. $\int \frac{t+1}{\sqrt{t}}\,dt$

4. $\frac{2}{3}t^{\frac{3}{2}} + 2t^{\frac{1}{2}} + c$

5. $\int (x^3 + e^{3x} + \ln(x))dx$

5. $\frac{1}{4}x^4 + \frac{1}{3}e^{3x} + x\ln x - x + c$

9. Integration by substitution

1. $\int (2x+1)(x^2+x+3)^5\ dx$
2. $\int (x-2)(x^2-4x+7)^3\ dx$
3. $\int x^2\sqrt{5x^3-6}\ dx$
4. $\int e^x(3e^x-1)^4\ dx$
5. $\int \frac{x}{(x^2-4)^2}\,dx$
6. $\int \frac{\ln x}{x}\,dx$
7. $\int \frac{2e^{5x}}{\sqrt{1-e^{5x}}}\,dx$

Answer

1. $\frac{1}{6}(x^2+x+3)^6 + c;$
2. $\frac{1}{8}(x^2-4x+7)^4 + c;$
3. $\frac{2}{45}(5x^3-6)^{3/2} + c;$
4. $\frac{1}{15}(3e^x-1)^5 + c;$
5. $-\frac{1}{2(x^2-4)} + c;$
6. $\frac{1}{2}(\ln x)^2 + c;$
7. $-\frac{4}{5}\sqrt{1-e^{5x}} + c.$

10. Logarithmic Integral:

1. $\int \frac{x}{x^2+1}\,dx$
2. $\int \frac{dx}{x\ln x}$
3. $\int \frac{3e^x}{2e^x+10}\,dx$
4. $\int \frac{x-1}{x^2-2x-6}\,dx$

Answer:

1. $\frac{1}{2}\ln(x^2+1) + c;$
2. $\ln(\ln x) + c;$
3. $\frac{3}{2}\ln(2e^x+10) + c;$
4. $\frac{1}{2}\ln(x^2-2x-6) + c$

11. The following all involve the "e" rule.

1. $\int e^{2x} dx$ — 1. $\frac{1}{2}e^{2x}+c;$

2. $\int 3.5e^{4.1x} dx$ — 2. $\frac{3.5}{4.1}e^{4.1x}+c;$

3. $\int xe^{x^2} dx$ — 3. $\frac{1}{2}e^{x^2}+c;$

4. $\int \frac{e^{\sqrt{x}}}{4\sqrt{x}} dx$ — 4. $\frac{1}{2}e^{\sqrt{x}}+c.$

12. Integration by parts — Answer

1. $\int xe^{x} dx$
2. $\int x\ln x \, dx$
3. $\int \ln x \, dx$
4. $\int x(1+x)^4 \, dx$
5. $\int x^2e^{x} dx$

Answer:

1. $xe^x - e^x + c$
2. $\frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$
3. $x\ln x - x + c;$
4. $\frac{x}{2}(1+x)^2 - \frac{1}{6}(1+x)^3 + c.$
5. $x^2e^x - 2xe^x + 2e^x + c$

13. Integration by substitution: — Answer:

1. $\int (x^2+1)(x^3+3x+5) \, dx$
2. $\int \frac{5x}{\sqrt[3]{x^2-8}} dx$
3. $\int \frac{e^x(x+1)}{xe^x} dx$
4. $\int \frac{\ln(x+1)}{x+1} dx$
5. $\int 4e^{2x}(3e^{2x}+5)^6 \, dx$
6. $\int 2x^2\sqrt{x^3+e} \, dx$
7. $\int \frac{\sqrt{\ln x}}{x} dx$

Answer:

1. $\frac{1}{6}(x^3+3x+5)^2+c;$
2. $\frac{15}{4}(x^2-8)^{2/3}+c;$
3. $\ln(xe^x)+c;$
4. $(\ln(x+1))^2+c;$
5. $\frac{2}{21}(3e^{2x}+5)^7+c;$
6. $\frac{4}{9}(x^3+e)^{3/2}+c;$
7. $\frac{2}{3}(\ln x)^{3/2}+c.$

14. Definite Integral: Answer

1. $\int_{1}^{2}(1+6x^2)dx = 15$ 2. $\int_{-2}^{-1}e^{2x}dx = \frac{1}{2}(e^{-2}-e^{-4})$ 3. $\int_{-1}^{1}xe^{x^2+1}dx = 0$ 4. $\int_{1}^{e}\ln x^2 dx = 2$

15. Area: Find the area of the following:

1. Bounded by x and y axis and $y = x^2 - 1$ Answer: 4/3

2. Enclosed by $y = -x$ and $y = 2 - x^2$ on [-1,3] Answer: 19/3

$$Solution: \int_{-1}^{2}(2-x^2-(x))dx + \int_{2}^{3}(-x-(2-x^2))dx$$

$$= \left(2x - \frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{2} + \left(-\frac{x^2}{2} - 2x + \frac{x^3}{3}\right)_{2}^{3}$$

$$= \left(4 - \frac{(2)^3}{3} + \frac{2^2}{2}\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) + \left(-\frac{3^2}{2} - 2(3) + \frac{3^3}{3}\right) - \left(-\frac{2^2}{2} - 2(2) + \frac{2^3}{3}\right)$$

$$= \frac{19}{3}$$

15. Application:

1. The daily profits in dollars of a firm is given by $P(x) = 100(-x^2 + 8x - 12)$, where x is the number of items sold. Find instantaneous rate of change where x = 3. Interpret your answer.

$P'(x) = -200x + 800$

$P'(3) = -200(3) + 800 = 200$

instantaneous rate of change=200 where x = 3.
That means marginal profit is 200

2. Algebraically find the following for the function $f(x) = 3x^5 - 20x^3 + 5$

a. All critical values

$f'(x) = 15x^4 - 60x^2$

$= 15x^2(x^2 - 4)$

$15x^2 = 0$ or $x^2 - 4 = 0$

$\therefore x = 0$ $\therefore x = \pm 2$

So, critical values are -2,0,2

b. the intervals where the function is increasing

Critical values -2 0 2

Test values		-3	-1	1	3
Sign of $f'(x)$	+	-	-	+	
$f(x)$	$(-\infty,-2)\uparrow$		$(-2,2)\downarrow$		$(2,\infty)\uparrow$

So, function increases on
$(-\infty,-2)\cup(2,\infty)$

c. the interval where the function is decreasing

Decreases at $(-2,2)$

d. the interval where the function is concave up

Inflection values		-1.41	0	1.41	
Test values		-2	-1	1	2
Sign of $f''(x)$	-	+	-	+	
$f(x)$	$(-\infty,-1.41)\cap$		$(-1.41,0)\cup$	$(0,1.41)\cap$	$(1.41,\infty)\cup$

concave up $(-1.41,0) and (1.41,\infty)$

e. the interval where the function is concave down

concave down $(-\infty,-1.41) and (0,1.41)$

f. all the inflection points

$$f''(x) = 60x^3 - 120x$$

$$= 60x(x^2 - 2)$$

$60x = 0$ or $x^2 - 2 = 0$

$\therefore x = 0$ $\qquad$ $\therefore x = \sqrt{2} = \pm 1.41$

3. Elasticity:

a) Consider the demand function $x = 30 - 2p$.

Then $\dfrac{dx}{dp} = -2$ $\qquad$ $p = \dfrac{30 - x}{2}$

Find the value of x for which the elasticity $E = 1$.

$$E = -\frac{p}{x}\frac{dx}{dp}$$

By plug in value of p, $E=1$ and derivative we get

$$1 = -\frac{30-x}{2x}(-2)$$
$$\Rightarrow x = 30 - x$$
$$\Rightarrow 2x = 30$$
$$\therefore x = \frac{30}{2} = 15$$

b) The weekly sales of Honolulu Red Oranges is given by $q = 936 - 18p$. Calculate the price elasticity of demand when the price is \$32 per orange. Also, calculate the price that gives a maximum weekly revenue. Answer 1.6, \$26

c) Suppose the likelihood that a child will attend a live musical performance can be modeled by $q = 0.01(0.0005x^2 + 0.39x + 33)$. $(15 \le x \le 100)$
Here, q is the fraction of children with annual household income x thousand dollars who will attend a live musical performance during the year. Compute the income elasticity of demand E at an income level of \$30,000. (Round your answer to two decimal places.) Answer: 0.28

d) The relation between traffic volume and expenditure of building roads are given as $T(K) = 0.4K^{1.06}$, K is expenditure. Find the elasticity of T with respect to K. Find also the consequences if expenditure increases by 10%.

Solution: $E = \frac{K}{T(K)}T'(x) = \frac{K}{0.4K^{1.06}} \times 0.4 \times 1.06K^{0.06} = 1.06$
The increase in 1% of expenditure would lead to 1.06% increase in traffic volume. The 10% increase in price would lead to a 10.6% increase in traffic volume.

4. Related Rates:

1. A car is traveling at 50 mph due south at a point 1/2 mile north of an intersection. A police car is traveling at 40 mph due west at a point 1/4 mile east of the same intersection. At that instant, the radar in the police car measurers the rate at which the distance between the cars is changing. What does the radar gun register?

2. A 10-foot ladder leans against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2 ft/s, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 ft off the ground?

3. The radius of a circular puddle is growing at a rate of 20 cm/s.
How fast is its area growing at the instant when the radius is 40 cm? (Round your answer to the nearest integer.). Answer: 5027 sq. cm/s

How fast is the area growing at the instant when it equals 25 cm^2? (Round your answer to the nearest integer.) Answer: 354 sq. cm/s

4. A rather flimsy spherical balloon is designed to pop at the instant its radius has reached 6 centimeters. Assuming the balloon is filled with helium at a rate of 17 cubic centimeters per second, calculate how fast the radius is growing at the instant it pops. Round your answer to two decimal places.) Answer: 0.04 cm/s

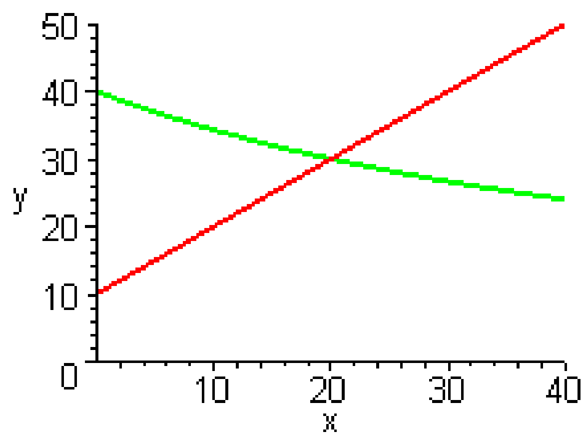
5. A circular conical vessel is being filled with ink at a rate of 20 cm^3/s. How fast is the level rising after 50 cm^3 have been poured in? The cone has a height of 60 cm and a radius 30 cm at its brim. (The volume of a cone of height *h* and cross-sectional radius *r* at its brim is given by $V = 1/3\, \pi r^2 h$) Answer: 0.77 cm/s

5. Consumers and producers surplus:

a) The demand and supply curves are $D(q) = q + 10$ and $S(q) = \dfrac{2400}{q+60}$. Find consumer's surplus and producer's surplus.

Solution. For equilibrium $D(q) = S(q) \Rightarrow q + 10 = \dfrac{2400}{q+60} \Rightarrow q = 20 = q_0$ and

$p_0 = D(q_0) = S(q_0) = 30$



Now PS = $p_0 q_0 - \int_0^{q_0} S(q)\,dq = 30 \cdot 20 - \int_0^{20} (q+10)\,dq = 600 - 400 = 200$

And CS = $\int_0^{q_0} D(q)\,dq - p_0 q_0 = 30 \cdot 20 - \int_0^{20} \dfrac{2400}{q+60}\,dq = 2400 \ln\left(\dfrac{80}{60}\right) - 600 = 90.44$